Decentralized Cooperative Stochastic Multiarmed Bandits

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Stochastic Multiarmed Bandits

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- The goal is to get cumulative reward as close as possible to the reward she could have obtained with the best fixed action, in hindsight.

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- In the stochastic version, rewards are sampled from a distribution which is arm-dependent. In the adversarial one, rewards are set by an adversary.
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In the stochastic case, in expectation, if Δ_k is the difference between the best mean and the mean for arm k:

$$R(T) = \mathbf{E}\left[\sum_{t=1}^{T} \Delta_{I_t}\right].$$

UCB

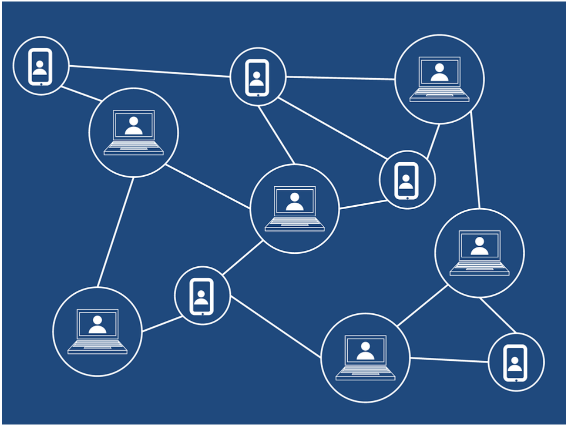
The UCB algorithm is an optimal algorithm for the multiarmed stochastic bandits problem. Fix a time step t and define:

- μ_t^k:= empirical mean observed for arm k.
- n_t^k:= number of times arm k was pulled.
- η:= exploration parameter > 1.
- I_t:= action played at time t.
- $UCB_k := \mu_t^k + \sqrt{\frac{2\eta\sigma^2 \ln t}{n_t^k}}$.

Then UCB picks the arm that maximizes UCB_k:

$$I_t = \mathrm{argmax}_{k \in [K]} UCB_k.$$

Decentralized Computation



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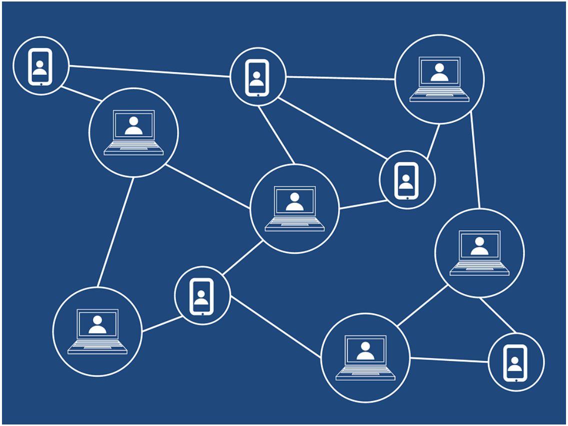

- The decentralization may be an **inherent restriction of the problem**, as is the case in packet routing or sensor networks, massive ad placement. It could be a choice made to **improve the total running time**.
- Focus should be graph-dependent algorithms, so we can cover the case in which we design the network and the case in which the network is a restriction.

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Adversarial case:

- **No communication case.** Regret incurred if separate optimal algorithms are run: $N\sqrt{KT}$.
- A lower bound: $N\sqrt{T}$.
- There is an algorithm achieving regret $N(\sqrt{K^{1/2}T\log K} + \sqrt{K}\log T)$ (**?**).

Gossip communication

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- Compute iteratively a weighted average of your value and the ones sent by your neighbors. If $x \in \mathbb{R}^N$ contains the values of the nodes, this operation is Px, for a *communication matrix* P. We just need P^s to converge to $\mathbb{1}\mathbb{1}^\top/N$.

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- Properties of P:
  - Supported on graph.
  - $P\mathbb{1} = \mathbb{1}; \qquad \mathbb{1}^\top P = \mathbb{1}^\top; \qquad |\lambda_2| < 1$

    $\iff P^s \xrightarrow[s\to\infty]{} \mathbb{1}\mathbb{1}^\top/N$ **(?)**
  - For acceleration eigenvalues must be real.

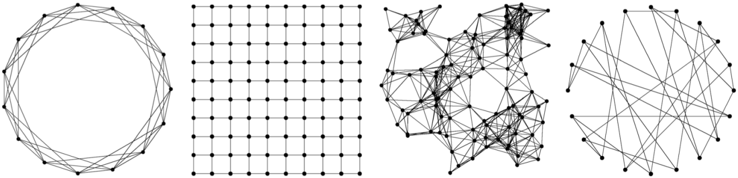
Gossip communication

Graphs with smaller spectral gap will benefit less from decentralization.

Example: time speedup of the dual averaging optimization algorithm with respect to non distributed computation is (**?**):

Graph	Cycle	Sq. Grid	Random Geometric Graph	Expander
Speedup	$\tilde{O}(1)$	$\tilde{O}(\sqrt{N})$	$\tilde{O}(\sqrt{N})$ *	$\tilde{O}(N)$

*Speedup with high probability for a random geometric graph on $[0,1]^2$ with connectivity radius $\Omega(\sqrt{\log^{1+\varepsilon} n/n})$ (for any $\varepsilon > 0$).



Model and Problem

- N agents in a graph with communication matrix P.
- Same K-armed bandit problem at each node.
- T time steps.
- Reward distributions are subgaussian with proxy σ^2 with means $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_K$.
- Regret is defined as

$$R(T) = TN\mu_1 - \mathbf{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}\mu_{I_{t,i}}\right] = \sum_{k=1}^{K}\Delta_k \mathbf{E}\left[n_T^k\right],$$

  The expectation is taken with respect to the algorithm and the rewards.
- poly(K) values are allowed to be communicated at each time step.
- We want the algorithm to be decentralized: only N and an upper bound on the spectral gap of P is assumed to be known to the agents, besides their corresponding row of P.

Decentralized UCB?

$$UCB_k := \mu_t^k + \sqrt{\frac{2\eta\sigma^2 \ln t}{n_t^k}}.$$

UCB needs to compute μ_t^k and n_t^k. In a decentralized setting could be approximated with gossip communication. It is an averaging problem.

Difficulties of the adaptation to the decentralized setting:

1. New rewards are obtained at each time step, the average is not only of one value per node.
2. We can only compute approximations of the averages.
3. It turns out that for getting good approximations we need to delay the rewards given to the UCB algorithm.

Running Consensus

Running Consensus (**?**). Values sequentially added to the network
$y_1, \ldots, y_t \in \mathbb{R}^N$ ($2K$ of these (μ_t^k and n_t^k for $k = 1, \ldots, K$))

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$$x_{t+1} = \underbrace{y_t + Py_{t-1} + \cdots}_{\text{NOT well-mixed}} + \underbrace{\cdots + P^{t-2}y_2 + P^{t-1}y_1}_{\text{well-mixed}}$$

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- Use rescaled Chebyshev polynomials.

Delayed average: a small-variance estimator

- If $C = \left\lceil \frac{\ln(N/\varepsilon)}{\ln(1/|\lambda_2|)} \right\rceil$ we have $\left\| P^C - \mathbb{1}^T \mathbb{1}/N \right\|_2 \leq \varepsilon/N$, (**?**).
- By rescaling Chebyshev's polynomials we can find the polynomial $q_s(x)$ of degree s with minimum supremum in the interval $[-|\lambda_2|, |\lambda_2|]$ that satisfies $q_s(1) = 1$, similarly as in (**?**). So $q_s(P)y_t$ can be computed with s gossip iterations.
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  - **Rewards obtained in the previous stage mix using the running consensus protocol.**
  - **Keep rewards of current stage in a third variable for mixing them in the next one.**
  - **At the end of the stage add the recently well mixed rewards to the rest of well mixed rewards.**

Pseudocode

Algorithm 1 Decentralized Delayed UCB at node i.

$\zeta \leftarrow$ Sample from each arm.
mixed$_i \leftarrow (\zeta/N, \mathbb{1}/N)$, mixing$_i \leftarrow (\zeta, \mathbb{1})$, new$_i \leftarrow \mathbf{0}$.
$t \leftarrow K, s \leftarrow K$
while $t \leq T$ **do**
  Obtain $\widehat{\mu}^k_{t,i}, \widehat{n}^k_{t,i}, k = 1, \ldots, K$ from mixed$_i$.
  $k^* \leftarrow \arg\max_{k \in \{1,\ldots,K\}} \left\{ \widehat{\mu}^k_{t,i} + \sqrt{\frac{2\eta\sigma^2 \ln s}{\widehat{n}^k_{t,i}}} \right\}$
  for C iterations **do**
    Perform one mixing step to mixing$_i$.
    Play arm k^*, update new$_i$.
    $t \leftarrow t + 1$
  end for
  $s \leftarrow (t - C)N$
  mixed$_i \leftarrow$ mixed$_i$+mixing$_i$.
  mixing$_i \leftarrow$ new$_i$.
  new$_i \leftarrow \mathbf{0}$.
end while

Regret

[M.R., Kanade, Rebeschini '19]

DDUCB incurs the following regret:

1. Accelerated communication:

$$R(T) \lesssim \sum_{k:\Delta_k>0} \frac{\sigma^2 \ln(TN)}{\Delta_k} + \frac{N \ln(N)}{\sqrt{\ln(1/|\lambda_2|)}} \sum_{k=1}^{K} \Delta_k.$$

2. Unaccelerated communication:

$$R(T) \lesssim \sum_{k:\Delta_k>0} \frac{\sigma^2 \ln(TN)}{\Delta_k} + \frac{N \ln(N)}{\ln(1/|\lambda_2|)} \sum_{k=1}^{K} \Delta_k.$$

The red term in both cases is NC, up to constants.
Acceleration needs less delay in the difficult regimes ($|\lambda_2|$ close to 1).

Interpretation of the regret

DDUCB (acceleration):

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Centralized algorithm

$$R(TN) \lesssim \sum_{k:\Delta_k>0} \frac{\sigma^2 \ln(TN)}{\Delta_k} + \sum_{k=1}^{K} \Delta_k.$$

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Lower bound (straighforward):

$$R(T) = \Omega\left(\sum_{k:\Delta_k>0} \frac{\sigma^2 \ln(TN)}{\Delta_k} + \left(\frac{N}{K} + 1\right) \sum_{k=1}^{K} \Delta_k\right)$$

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DDUCB without hiding the stages hyperparameter ε and the exploration parameter η:

$$R(T) \lesssim \sum_{k:\Delta_k>0} \frac{\eta(1+\varepsilon)\sigma^2 \ln(TN)}{\Delta_k} + \left(\frac{N\ln(N/\varepsilon)}{\ln(1/|\lambda_2|)} + \frac{\eta}{\eta-1}\right) \sum_{k=1}^{K} \Delta_k.$$

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Algorithm in (**?**), coopUCB:

$$R(T) \lesssim \sum_{k:\Delta_k>0} \sum_{j=1}^{N} \frac{\gamma(1+\varepsilon_c^j)}{N\Delta_k} \ln(TN) + N\left(\sqrt{N}\sum_{j=2}^{N} \frac{|\lambda_j|}{1-|\lambda_j|} + \frac{\gamma}{\gamma-1}\right) \sum_{k=1}^{K} \Delta_k$$

Comparison with previous work

- **?** designed and analyzed coopUCB.
- coopUCB runs a decentralized UCB but with no delays.
- Size of confidence intervals increases to compensate for the inaccuracy of the estimation.
- It needs the knowledge of more data about the graph (whole spectrum of P and its eigenvectors).

Comparison with previous work

For coopUCB, the algorithm in (?) the regret is:

$$R(T) \lesssim \alpha \sum_{k:\Delta_k>0} \frac{\sigma^2 \log TN}{\Delta_k} + \beta \sum_{k=1}^{K} \Delta_k$$

and

$$1 \lesssim \alpha \quad ; \quad \frac{N \ln(N)}{\ln(1/|\lambda_2|)} \lesssim \beta.$$

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- Communication is trivial in the case of a complete graph. It is a batched bandit problem.

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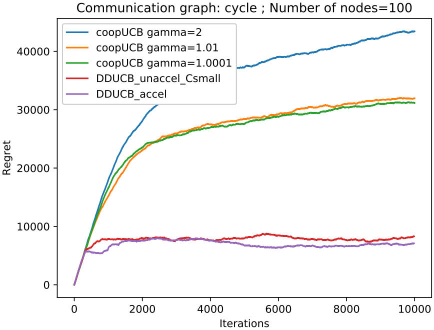
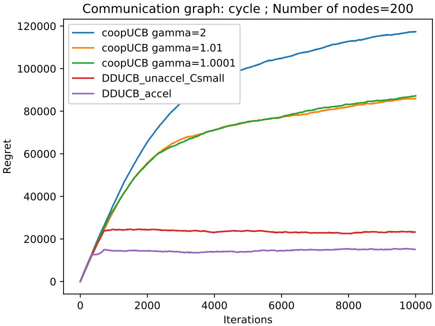
Example:

Graph: Cycle	α	β
[Landgren et al. '16]	$\Theta(N^2)$	$\Theta(N^{7/2})$
DDUCB (unacc)	$\Theta(1)$	$\Theta(N^3 \log(N))$
DDUCB (acc)	$\Theta(1)$	$\Theta(N^2 \log(N))$

Simulations





Cycle with $N = 100$ (left) and $N = 200$ (right).

$K = 17$; optimal arm $\mathcal{N}(1, 1)$; sub-optimal arms $\mathcal{N}(0.8, 1)$

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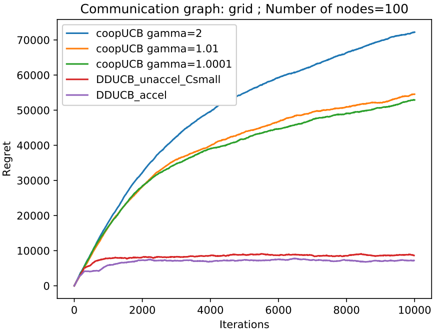

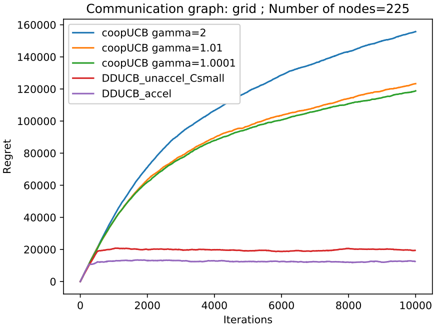

Square grid with $N = 100$ (left) and $N = 225$ (right).

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  - This yields $\mathbf{P}\left(I_{t,v} = k \,|\, n_{t,v}^k > \frac{16\eta\sigma^2 \ln(N d_t)}{\Delta_k^2}\right) \leq \frac{2}{(N d_t)^\eta}$

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5. Reduce communication at the expense of increasing delay: send the same information in more steps. (Analysis is straightforward since we worked with an arbitrary fixed delay).

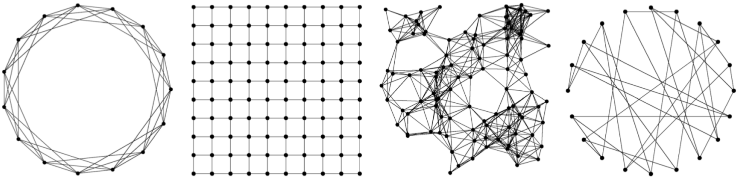
Spectral gap and regret

Graphs with greater spectral gap benefit more from decentralization.

Terms of the regret for DDUCB, if regret is
$R(T) \lesssim \alpha \sum_{k:\Delta_k>0} \frac{\sigma^2 \log TN}{\Delta_k} + \beta \sum_{k=1}^{K} \Delta_k$

Graph	Cycle	Square Grid	Random Geom. Graph	Expander
Samples per it.	N	N	N	N
α	$O(1)$	$O(1)$	$O(1)$	$O(1)$
β	$\tilde{O}(N^2)$	$\tilde{O}(N^{3/2})$	$\tilde{O}(N^{3/2})$ *	$\tilde{O}(N)$

*Second term of regret with high probability for a random geometric graph on $[0,1]^2$ with connectivity radius $\Omega(\sqrt{\log^{1+\varepsilon} n/n})$ (for any $\varepsilon > 0$).



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- First algorithm with size-independent $\log T$-term in regret.
- Communication can be accelerated using Chebyshev polynomials.
- **Trade-off** between regret and communication (graph & delay decisions).