Cheap Orthogonal Constraints in Neural Networks: A Simple Parametrization of the Orthogonal and Unitary Group

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We study the optimization of neural networks with orthogonal constraints

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  - They allow for the implementation of factorized linear layers.

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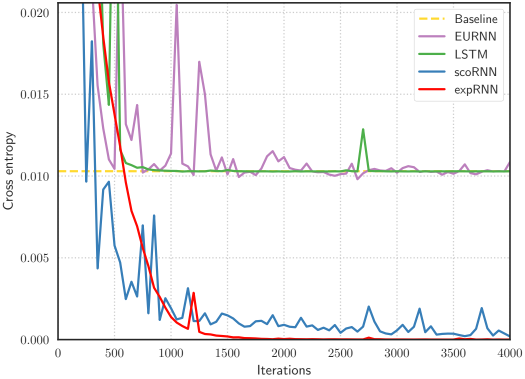
  - General purpose optimizers can be used (SGD, ADAM, ADAGRAD, ...).
  - **No new extremal points** are created in the main parametrization region.



Cross entropy in the copying problem for $L = 2000$.

The copying problem uses synthetic data of the form:

	Random numbers	Wait for L steps	Recall
Input:	14221	-------	:-----
Output:	-----	-------	14221

MODEL	N	# PARAM	VALID.	TEST
EXPRNN	224	$\approx 83K$	5.34	5.30
EXPRNN	322	$\approx 135K$	**4.42**	**4.38**
EXPRNN	425	$\approx 200K$	5.52	5.48
SCORNN	224	$\approx 83K$	9.26	8.50
SCORNN	322	$\approx 135K$	8.48	7.82
SCORNN	425	$\approx 200K$	7.97	7.36
LSTM	84	$\approx 83K$	15.42	14.30
LSTM	120	$\approx 135K$	13.93	12.95
LSTM	158	$\approx 200K$	13.66	12.62
EURNN	158	$\approx 83K$	15.57	18.51
EURNN	256	$\approx 135K$	15.90	15.31
EURNN	378	$\approx 200K$	16.00	15.15
RGD	128	$\approx 83K$	15.07	14.58
RGD	192	$\approx 135K$	15.10	14.50
RGD	256	$\approx 200K$	14.96	14.69

RNNs trained on a speech prediction task on the TIMIT dataset.
It shows the best validation MSE accuracy.